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A user's guide: Monoidal Bousfield localizations and algebras over operads

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4. Colloquial summary

Mathematics at its most basic level is the study of abstract thinking. Category theory follows this approach, and interprets all branches of mathematics as the study of objects and the relationships between them. For example, the objects might be people and you might say two are related if they're friends, or if one follows the other's Twitter feed, or if they were in the same graduating class of high school, etc. Or the objects might be real numbers, and you might decide one is related to another if it's smaller. Or the objects might be shapes, and you could say two are related if they have the same symmetries (so a pentagon and a 5-pointed star are related, but the pentagon and the square are not related). When two objects are so tightly related that we want to view them as the same in all settings then we say those objects are equivalent (mathematicians use the term *isomorphic*). For example, we might reasonably decide that all squares of side length 2 inches are the same. So we can make a category of squares, and if two different squares have side length 2 inches, they are equivalent in our category. On the other hand, maybe someone else wants to study squares that come equipped with a color, so that red squares can be distinguished from blue squares. That's a different category, in which two squares with the same color and same side length would be considered equivalent. All of the work in this paper is in the setting of category theory, but to understand it we must now introduce another player.

Localization is a fundamental tool in mathematics that allows one to zoom in on the pertinent information in a problem. In the context of category theory, localization is a way to view two different (i.e. non-equivalent) objects as equivalent, e.g. deciding we don't care any more about color and now a red square and a blue square can be equivalent if they have the same side length. What's happening here is that we are putting on different eyeglasses when looking at the objects we want to study. Mathematically it means we're allowing more relationships between the objects, e.g. allowing blue and red to be related when before they were not. As humans we do this all the time. For example, if two driving routes take the same amount of time we might view them as equivalent. If two types of pasta in the supermarket cost the same then we might view them as the same (if we don't care that much about pasta), and that's a valuable way to focus in on the information (in this case, price) that really matters to us.

In order to best study this localization procedure, we work in the setting of special kinds of categories called model categories. These are categories that come equipped with a specific localization we plan to do but have not done yet. This means there is a specific collection of objects we want to force to be equivalent. More formally, it means there is a specific collection of new relationships we want to add to the category so that the objects they are relating become "the same." These relationships are called *weak equivalences* because they are not equivalences yet. For example, our category might be the category whose objects are shapes of all colors and where one shape is related to a second shape if they have the same color and if all the symmetries of the first are also symmetries of the second (e.g. a triangle is related to a hexagon). Then two objects are going to turn out to be equivalent if they have the same shape, size, and color. The weak equivalences could be relationships that ignore side length, so that two objects are weakly equivalent if they have the same shape and color, but not necessarily the same size.

Model categories admit a special kind of localization called Bousfield localization (named after the mathematician Pete Bousfield) that transforms a model category into another model category with even more weak equivalences (i.e. we plan to view even more objects as equivalent). This procedure sends every object X to a closely related one (we'll denote it L(X)) that is equivalent to the original object according to the new notion of equivalence, but not according to the old notion. In our example above of shapes, sizes, and colors the new weak equivalences could be maps where the symmetries of one shape are symmetries of the second, but with no mention of side length or color (so we've added more relationships). The result is that two objects are weakly equivalent according to the new weak equivalences if they have the same shape, but not necessarily the same size or color.

This paper is fundamentally about Bousfield localization. I studied how much structure on an object X is destroyed by the passage to L(X). Specifically, I was interested in algebraic structure on X. To a mathematician, algebra is a powerful computational tool and a great way of determining whether two objects are equivalent or not. Algebraic structure on an object should be thought of as icing on a cake. If two cakes have different icing then that's one sure-fire way to know they are different. Going back to our example of shapes, the information regarding the symmetries of the shape can be viewed as algebraic structure. One can describe a shape by the number of sides it has and where it's located in space, but this information about symmetries is extra and is often very useful. It's one way we can tell two triangles apart, for example (e.g. if one is equilateral and the other is obtuse).

One way to encode algebraic structure in a category (i.e. to allow objects to possess algebraic structure) is via gizmos called *operads*. For a given type of algebraic structure you want to study (e.g. a flavor of icing on the cake) these gizmos tell you exactly which objects have that structure. If you've got an object X that has the structure it's a natural question to ask whether its localization L(X) still has that structure. This paper answers that question in general by writing down exactly what must be satisfied in order for the algebraic structure to be preserved. It then goes on to work out specifically what that answer means in a number of model categories that people have studied. I want to pause for a moment to explain why it was important to work out these specific cases of the general result.

Perhaps unsurprisingly, not all mathematicians like category theory. I love it, because if you can prove a theorem in category theory then it's true in all branches of mathematics. Similarly, if you prove a result about model categories then it's true in every specific model category out there. However, if someone is working in a specific branch of math (e.g. geometry) and the theorem they were trying to prove is proven using category theory they might be understandably frustrated; it feels like cheating. The relationship between category theory and the rest of math is much like the relationship between math and the rest of science. That is to say: category theorists produce results which can help in all areas of math, if someone takes the time to translate from the category theoretic jargon into the language folks in those areas are used to. Similarly, mathematics produces results useful all over science: both in physical science and social science, but often researchers in those fields don't want to go and learn a bunch of math in order to understand what has been done, so it falls to interdisciplinary researchers (e.g. mathematical physicists, mathematical biologists, mathematical economists, etc.) to bridge the gap and translate these general results into specific results in those fields. It often happens in economics that a team consisting of a mathematician and an economist are jointly awarded a Nobel Prize, because the mathematician worked out the general theory while the economist found lots of real-world applications for it. I think the analogy can be taken one step further. The relationship between a mathematician and a general scientist is like the relationship between the person who writes a cookbook and the actual cook in the kitchen. Alone neither might be successful but together they can produce something to better the world. If cookbook authors went around using all sorts of jargon that chefs could not recognize (e.g. discussing the use of "positively curved metallic tools controlled via torque" instead of calling them "spoons") they'd be doing their own work a disservice.

In my case I wanted people to use my work. So I learned the specific properties of a number of different model categories, and in each case I made sure the hypotheses for my general result held in those settings. This way I knew my work applied to a number of subfields of math, including algebra (the study of algebraic structure), topology (the study of space), (stable) homotopy theory (the study of when two spaces can be continuously deformed to become equivalent), representation theory (a way of studying objects based on how they cause other objects to change), and even to category theory itself. In each of these settings I learned the domain-specific jargon and proceeded to state my main result in that language, in the hopes that researchers in those fields would be able to use my work and would be comfortable doing so.

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