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A user's guide: A monoidal model for Goodwillie derivatives

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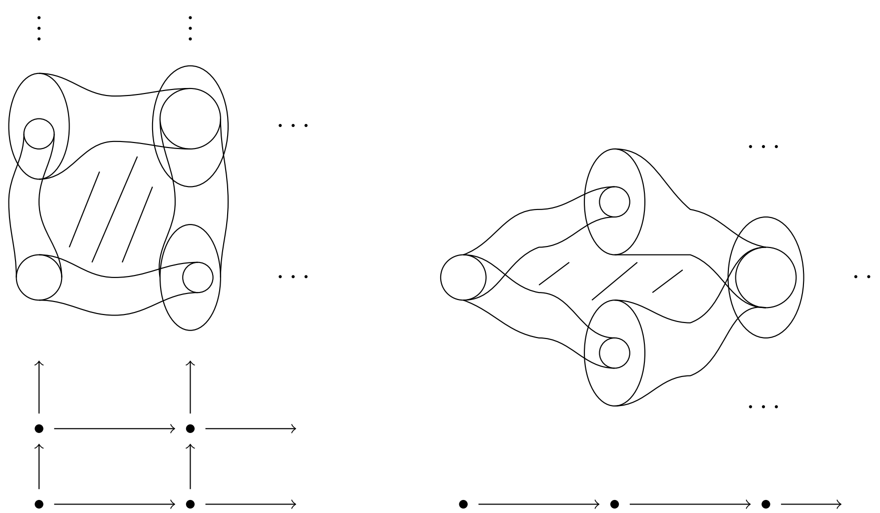
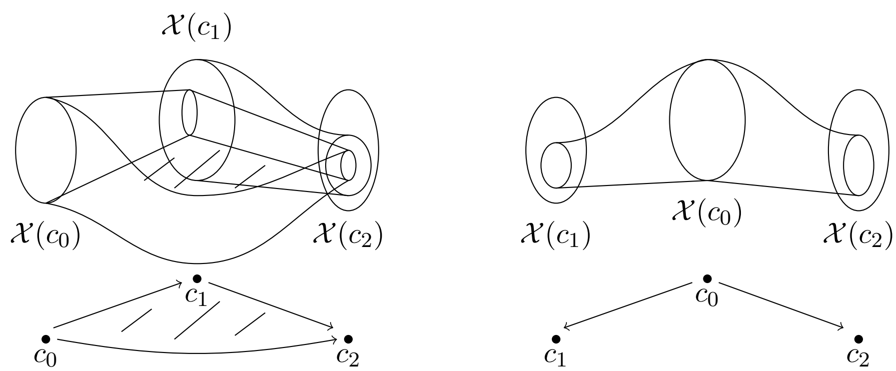


2. Metaphors and imagery

We will use this section to give some intuition for homotopy colimits and what "combining" them entails. We will discuss why we should believe that the derivatives that show up in functor calculus are multilinearizations of cross effects, and, as foretold, we will describe the sphere operad and the salient properties that allow for monoidal derivatives.

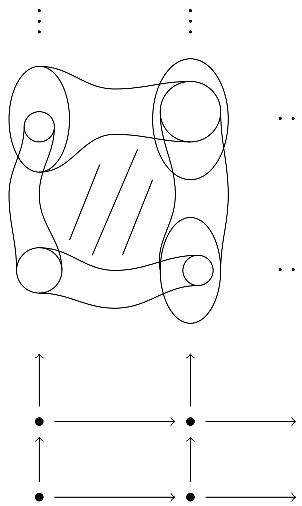
2.1. Homotopy colimits. The colimit of a diagram is an object that encodes the limiting behavior of maps, and one can think of the homotopy colimit as physically keeping track of the maps along the way. Bousfield and Kan give a description of the homotopy colimit as the geometric realization of a simplicial set indexed by the nerve of the diagram category; for a diagram $\mathcal{X} : \mathcal{C} \to \mathcal{T}$, the n-simplices of the homotopy colimit $\operatorname{hocolim}_{\mathcal{C}} \mathcal{X}$ are $\coprod_{c_0 \to \cdots \to c_n} \mathcal{X}(c_0)$.

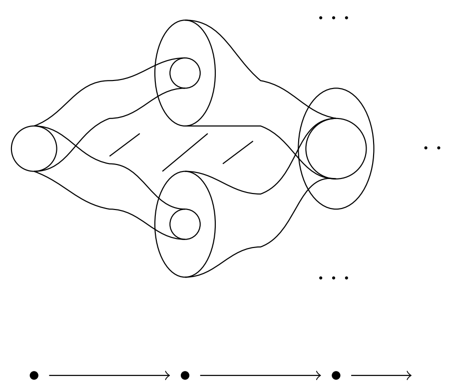
Upon realization, each object of the diagram gets a space, each arrow of the diagram gets a copy of the source space times a 1-cell that traces out the map between the corresponding source and target spaces, each pair of composable morphisms gets the source space times a 2-cell which fills in the triangle formed by the composite 1-cells, etc. Two basic diagram shape examples are given below, shown with the homotopy colimits living above their diagrams.



These images are not new to anyone familiar with Dugger's excellent primer [**Dug**], and I strongly recommend it for more information. One can imagine the homotopy colimit space as living above the diagram shape category $\mathcal{C}$, and this gives a bit more intuition when it comes to visualizing maps between homotopy colimits resulting from changing the diagram shape. We will use this imagery to describe the "combining" of homotopy colimits mentioned in Topic 1. Since Bousfield-Kan's description of the homotopy colimit [**BK72**] is the realization of a simplicial set indexed on the nerve of the diagram shape category $\mathcal{C}$, we can think of individual n-cells of that simplicial set as $(x, c_0 \to \cdots \to c_n) \in \mathcal{X}(c_0) \times N_n\mathcal{C}$. In the figure above, for example, a 1-cell is a point in one of the spaces along with its image under an arrow.

An important fact of homotopy colimits is that when there is a map between diagram categories $\alpha : \mathcal{C} \to \mathcal{D}$, there is an induced map between the homotopy colimits indexed over those categories, $\operatorname{hocolim}_{\mathcal{C}} \mathcal{X} \circ \alpha \to \operatorname{hocolim}_{\mathcal{D}} \mathcal{X}$. The simplex $(x; c_0 \to \cdots \to c_n)$ with $x \in \mathcal{X}(\alpha(c_0))$ is sent to the simplex $(x; \alpha(c_0) \to \cdots \to \alpha(c_n))$. A relevant example is the map of diagram categories $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by addition. On homotopy colimits, this yields a map $\operatorname{hocolim}_{\mathbb{N}\times\mathbb{N}} \mathcal{X} \circ \alpha \to \operatorname{hocolim}_{\mathbb{N}} \mathcal{X}$. A (truncated) picture is given below, where the first image is a homotopy colimit over $\mathbb{N} \times \mathbb{N}$ and the second is its image under α.





This is what we mean by combining homotopy colimits; the diagram map $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ given by addition yields a map on homotopy colimits that is associative up to homotopy, but not strictly associative. The problem is similar to that for finding a strictly associative smash product for spectra; there is essentially an $\mathbb{N}\times\mathbb{N}$ grid of spaces (each spectrum has $\mathbb{N}$ spaces, but with structure maps a little different than the grid above) and the goal is to combine this information into only $\mathbb{N}$ spaces. This is possible by choosing a path through the grid, but this is only associative up to homotopy. To make the combination strictly associative, one thing that works for the smash product is to introduce symmetric group actions into the objects of the grid. We are essentially doing the same thing by including all injections of finite sets into the diagram shape, changing the diagram from $\mathbb{N}$ to $\mathbb{I}$. But how does including symmetry help? For the smash product, including the symmetry is like including all possible paths; for the colimits, the extra maps encoding the symmetry make the homotopy colimits enormous, so that they contain all possible ways you could squish the simplices over to one copy of $\mathbb{I}$.

2.2. Function derivatives to Goodwillie derivatives. While the Goodwillie-Taylor tower and the layers have natural analogs in the function calculus world, the literature does not have a description of why the multilinearized cross effects make any sense as derivatives. This yields a fun exercise in the analogies of calculus, so we include it here.

We will start with what we know from function calculus, the first derivative of a function f at 0,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}.$$

We use the standard dictionary to get to functor calculus, so replace x with space X, 0 with the 0-space $*$, and subtraction with homotopy fiber. We also note that the numerator is a familiar object, the first cross-effect.

$$f'(*) = \lim_{X \to *} \frac{\mathrm{hofib}[f(X) \to f(*)]}{\mathrm{hofib}[X \to *]} \simeq \lim_{X \to *} \frac{cr_1 f(X)}{X}$$

To take a limit as a space approaches another space, we need to talk about the “topology” on the category of spaces. The idea is to think of connectivity as giving distance, so being more highly connected means being closer to the one-point space. One way to take a limit as x approaches a is to look at a sequence converging to a, so we choose the convenient sequence of spheres, $S^0, S^1, S^2, \ldots$ which ‘converges’ to the contractible S^∞. Thus our limit becomes

$$f'(*) \simeq \lim_{n \to \infty} \frac{cr_1 f(S^n)}{S^n} \simeq \lim_{n \to \infty} \Omega^n cr_1 f(S^n).$$

The final equivalence is thinking of looping as the inverse of suspending. It’s not a perfect dictionary, but this is the linearization of the first cross effect, as promised. Then the higher derivatives are iterates of this. Recall that the second cross effect is a total homotopy fiber of a square that can be computed with iterative fibers.

$$\begin{tikzcd}
cr_2 F(X,Z) \arrow[d, dotted] & & \\
{[F(X \vee Z) - F(X)]} \arrow[r, dashed] \arrow[d, dotted] & F(X \vee Z) \arrow[r] \arrow[d] & F(X) \arrow[d] \\
{[F(Z) - F(*)]} \arrow[r, dashed] & F(Z) \arrow[r] & F(*)
\end{tikzcd}$$

Cross effects existed before functor calculus, and the second cross effect $cr_2 f$ is a function of two variables defined by $cr_2 f(x,z) = [f(x+z) - f(x)] - [f(z) - f(0)]$.

Then the second derivative at zero is the following.

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0} \frac{\lim_{y \to x} \frac{f(y) - f(x)}{y - x} - \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}}{x - 0}$$

Replace y with $x + z$ and replace the limit over y with a limit as z goes to zero. In our translation, addition is replaced by wedge sum, so we have

$$f''(*) \simeq \lim_{X \to *} \lim_{Z \to *} \frac{f(X \vee Z) - f(X) - f(Z) + f(*)}{XZ} \simeq \lim_{n \to \infty} \lim_{k \to \infty} \Omega^n \Omega^k cr_2 f(S^n, S^k)$$

the multilinearized second cross effect.

2.3. The sphere operad. We mentioned that the sphere operad of [**AK14**] is a vital component of the definition of monoidal derivatives, and we will describe why it is necessary and how one can think of it. Recall that a goal is to define a map $\partial_1 F \wedge \partial_2 G \to \partial_2(F \circ G)$. As defined in the paper, $\partial_1 F \wedge \partial_2 G$ is

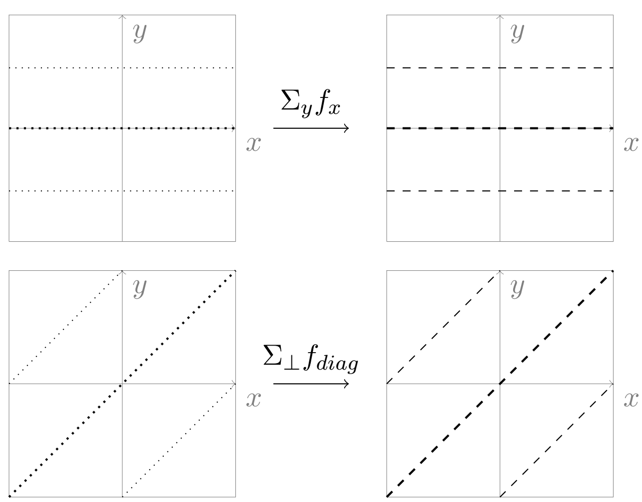
$$\operatorname*{hocolim}_{n} \Omega^n cr_1 F(S^n) \wedge \operatorname*{hocolim}_{k,\ell} \Omega^k \Omega^\ell cr_2 G(S^k, S^\ell).$$

The idea is to double up the n-coordinates so that once the second cross effect is assembled into the first, the S^n can assemble into both variables of $cr_2 G$. This requires a map $\Omega^n H(S^n) \to \Omega^n \Omega^n H(S^n \wedge S^n)$ which is associative and equivariant (with respect to swapping the two copies of S^n). It's easier to envision this if H is the identity functor.

Let S^n be the one-point compactification of $\mathbb{R}^n$, and consider an element f of ΩS^1, a map $f : S^1 \to S^1$. To get a map $S^2 \to S^2$, we could use $S^1 \wedge f$, but this new element of $\Omega^2 S^2$ needs to have a trivial Σ_2-action, so must be symmetric on the two sphere coordinates, not just the identity on one. One fix is to put the element f in as the diagonal in $\mathbb{R}^2$ and suspend it in the antidiagonal direction. That is, given a map

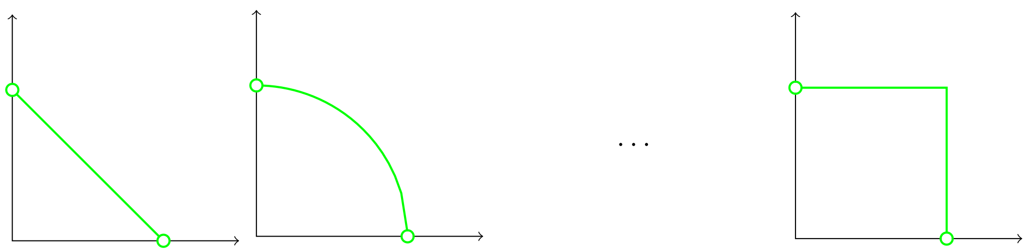
$$f : \cdots\cdots \to - - - - -$$

the options are to think of this sphere as one of the coordinates of S^2 while suspending in the direction of the other coordinate, or to think of it as the diagonal in S^2 while suspending in the perpendicular direction (these two options are pictured below).



The suspension along the second coordinate, depicted in the top picture, is clearly not equivariant with respect to permuting coordinates; the second suspension, while equivariant in the appropriate way, is not strictly associative. This can be seen by considering the suspension that would define maps $\Omega S^1 \to \Omega^k S^k$ for higher k. This is where the sphere operad comes into play. We are essentially finding a "complement" sphere to suspend along; in the first example, we've chosen one along the vertical axis, and in the second example, the complement runs along the antidiagonal $y = -x$. We will describe the sphere operad and how its components give good complements.

First, we describe the nonunital simplex operad, whose nth space is an open $n-1$-dimensional simplex. In level n, it is the limit over k of the n-simplices in $\mathbb{R}^{n+1}$ described by $x_1^k + \cdots + x_n^k = 1$ with $x_i > 0$. In level 2, this looks something like the final image below.



$x_1 + x_2 = 1$ $\quad$ $x_1^2 + x_2^2 = 1$ $\quad$ $x_1^k + x_2^k = 1$ $\quad$ $x_1^\infty + x_2^\infty = 1$

The sphere operad $\mathbf{S}$ is the (levelwise) one-point compactification of the simplex operad, so the nth space of $\mathbf{S}$ is homeomorphic to S^{n-1}. The operad composition maps are homeomorphisms

$$S^{k-1} \wedge S^{j_1-1} \wedge \cdots \wedge S^{j_k-1} \to S^{j_1+\cdots+j_k-1}.$$

There is a map of operads $\mathbf{S} \to \mathrm{Coend}(S^1)$ such that for each $n \geq 1$ the map $\mathbf{S}_n = S^{n-1} \to \Omega S^n$ is adjoint to a homeomorphism $S^{n-1} \wedge S^1 \to S^n$. Since the Σ_n-action on the coendomorphism operad of S^1 permutes the n coordinates of S^n, this defines a Σ_n-equivariant map $S^1 \wedge \mathbf{S}_n \cong S^n$. Finally, there is a map of operads $Com \to \mathbf{S}$ such that the composite $Com \to \mathbf{S} \to \mathrm{Coend}(S^1)$ is levelwise the canonical map adjoint to the diagonal map $S^1 \to S^n$.

These are the properties that ensure the equivariance that we need. In our basic example $\Omega S^1 \to \Omega^2 S^2$, the desired map puts f into $\mathbb{R}^2$ on the diagonal, but now suspends along $\mathbf{S}_2$ instead of the antidiagonal. The fact that $\mathbf{S}_k$ assemble into an operad means that we will also get the associativity that we need.

References

[AK14] Gregory Arone and Marja Kankaanrinta, *The sphere operad*, Bull. Lond. Math. Soc. **46** (2014), no. 1, 126–132.

[BK72] A.K. Bousfield and D.M. Kan, *Homotopy Limits, Completions and Localization*, Lecture Notes in Mathematics, vol. 304, Springer-Verlag Berlin Heidelberg, 1972.

[Dug] Dan Dugger, *A primer on homotopy colimits*, http://pages.uoregon.edu/ddugger/hocolim.pdf.

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