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A user's guide: A monoidal model for Goodwillie derivatives

Sarah Yeakel

1. Key insights and central organizing principles

1.1. The question. In order to understand certain functors, Goodwillie developed a theory of polynomial approximations for homotopy invariant functors from pointed topological spaces $\mathcal{T}$ to the categories of spaces $\mathcal{T}$ or spectra $\mathcal{S}\mathrm{p}$ in a series of papers called Calculus I, II, and III [**Goo90**, **Goo91**, **Goo03**], a nod to the Taylor series approximation method of function calculus.

He describes a way to canonically assign to a functor F a sequence of "polynomial" (called *n-excisive*) functors P_nF approximating F, which fit into a tower of fibrations analogous to a Taylor series. Thus this tower of functors along with the natural maps $p_nF : F \to P_nF$ are called the Taylor tower for F:

$$\begin{tikzcd}
F(X) \arrow[dr, bend right] \arrow[drr, bend left] & & & & & \\
\cdots \arrow[r] & P_nF(X) \arrow[r] & P_{n-1}F(X) \arrow[r] & \cdots \arrow[r] & P_1F(X) \arrow[r] & P_0F(X) \simeq F(*)
\end{tikzcd}$$

As in function calculus, one tries to study the functor F by studying its Taylor tower, and this is a good approximation when F is *analytic*, which implies $F(X) \simeq \operatorname{holim}_n P_nF(X)$ for sufficiently connected X; we call this convergence in a radius of convergence. Many functors are analytic; some examples include the identity functor of spaces, the functor $F(X) = \Sigma^\infty X^{\wedge k}$, and the representable functors $F(X) = \operatorname{Hom}(K, X)$ for finite dimensional spaces K.

While the analogy with function calculus is fun and illuminating, there is an immediate roadblock to understanding functor calculus with this approach because the n-excisive approximations, P_nF, are difficult to compute; for example, the first approximation of the identity functor is $P_1Id(X) \simeq \Omega^\infty\Sigma^\infty X$, the stable homotopy functor. Since the Taylor tower of the identity functor converges (on appropriately connected spaces), we can view the levels P_nF as interpolating between stable (at level 1) and unstable (in the limit) homotopy theory, which is all rather nontrivial to compute for many spaces.

Because the levels of the Taylor tower are computationally opaque, we turn to the homotopy fibers $D_nF = \mathrm{fiber}(P_nF \to P_{n-1}F)$, or *layers*, of the Taylor tower, with the hopes that the polynomial parts can be reconstructed once the layers are known.

One can think of the fiber D_nF as a difference of the nth polynomial approximation from the $n-1$st; in function calculus, this procedure would produce the n-homogeneous functions $D_nf = \frac{f^{(n)}(0)\cdot x^n}{n!}$. Indeed, the analogy is again justified by Goodwillie's classification of the layers.

Theorem 1.1 ([**Goo03**]).

$$D_nF(X) \simeq \Omega^\infty \left(\partial_nF \wedge X^{\wedge n}\right)_{h\Sigma_n}$$

where ∂_nF is a spectrum with Σ_n-action called the n-th derivative of F.

The symmetric group Σ_n acts on the smash product by permuting the factors and $(-)_{h\Sigma_n}$ denotes the homotopy orbits, which could be interpreted as dividing by $n!$. Taken together, the derivatives form a symmetric sequence in the category of spectra, and we see that the layers of the tower are determined by ∂_*F. This means that perhaps the extension problems of recovering the levels of the Taylor tower from the layers could be solved by exploring extra structure in the derivatives. Indeed, the derivatives are rich in structure; it should be noted that many people have worked on this problem, to much avail. As a start, Goodwillie identifies the homotopy type.

Theorem 1.2 ([**Goo03**]). *The n-th derivative of F is equivalent to the multilinearization of the nth cross-effect.*

$$(\Omega^\infty)\partial_nF \simeq \underset{k_1,\ldots,k_n\to\infty}{\mathrm{hocolim}}\ \Omega^{k_1}\cdots\Omega^{k_n} cr_nF(\Sigma^{k_1}S^0,\ldots,\Sigma^{k_n}S^0)$$

The Σ_n-action is induced by permuting the variables of cr_nF; in the multilinearization, this also permutes the loops. The nth cross effect is a functor of n variables which can be thought of as a measurement of the failure of F to be degree $n-1$ (in a sense). For example, $cr_1F(X) = \mathrm{hofib}(F(X) \to F(*))$, so if F is degree 0 (or constant), cr_1F is trivial. The higher cross effects are defined as total homotopy fibers of cubes, for example, cr_2F is a functor of two variables, defined as the total fiber of the following 2-cube.

$$\begin{tikzcd} F(X \vee Y) \arrow[r] \arrow[d] & F(X) \arrow[d] \\ F(Y) \arrow[r] & F(*) \end{tikzcd}$$

Considering all the derivatives of a functor at the same time yields a symmetric sequence in spectra. Thus we may think of the derivatives as a functor

$$\partial_* : \mathrm{Fun}(\mathcal{T}, \mathcal{T}) \to \mathrm{Fun}(\Sigma, \mathcal{S}\mathrm{p}).$$

Both the domain and codomain have monoidal structures; in the domain, it's composition of functors, in the codomain, the circle product of symmetric sequences. This point of view leads to a question posed by Arone and Ching in the introduction of [**AC11**], which is the main goal of [**Yea17**].

QUESTION 1.1. *Is ∂_* monoidal?*

Since monoidal functors preserve monoids, this would imply that the derivatives of monads (which are monoids in the category of endofunctors) on topological spaces are operads (which are the monoids under the circle product of symmetric sequences). Then the derivatives of the identity functor would form an operad, and derivatives of other functors would be modules over that operad. This is the kind of extra structure we are looking for, although Arone and Ching have shown [**AC11**] that it is not enough to recover the levels of the Taylor tower. In nontechnical terms, being monoidal says that the derivatives of functors behave well with respect to composition and the derivatives of the identity functor fit together with themselves and with other derivatives in interesting ways.

Specifically, being monoidal would require a natural transformation $\partial_* F \circ \partial_* G \to \partial_*(F \circ G)$ and a map $\mathbb{S} \to \partial_1 Id$, where the first $\circ$ is the circle product of symmetric sequences and the second is composition of functors. These maps should fit into commutative diagrams for commutativity, unitality, and equivariance. The first map looks like a sort of chain rule for functors, describing a relationship between the derivative of a composition and the derivatives of the composites.

In [**Yea17**], we describe a model for the derivatives which is monoidal, giving a positive answer to the question of Arone and Ching above. Verifying that all the necessary diagrams commute is part of the drama, and we will now relive some of the obstacles encountered, drawing attention to some important concepts.

1.2. The hurdles. There are some drawbacks to working with Goodwillie's model for the derivatives. By universal properties and choosing models for homotopy (co)limits, it seems that a map on compositions $\mu : \partial_* F \circ \partial_* G \to \partial_*(F \circ G)$ is not hard to construct, but such a map would not necessarily be strictly associative. That is, one could easily define a map μ which is associative and unital up to homotopy, but strict associativity requires a different model for the derivatives. While frustrating, this leads to a very important theme of the paper.

KEY IDEA 1.2. *The nitpicky details are important; be careful choosing models for homotopy functors.*

A map of symmetric sequences is a levelwise morphism for each natural number. In level n, μ takes the form

$$\bigvee_{j_1+\cdots+j_k=n} \partial_k F \wedge \partial_{j_1} G \wedge \cdots \wedge \partial_{j_k} G \longrightarrow \partial_n(F \circ G).$$

A map out of a coproduct can be defined by a product of maps out of the summands, and some of these summands, for example when $n = 2$, look like $\partial_1 F \wedge \partial_2 G$, $\partial_2 F \wedge \partial_1 G \wedge \partial_1 G$, or $\partial_2 F \wedge \partial_0 G \wedge \partial_2 G$ (the coproduct is enormous). In the paper, we focus on *reduced* functors, those which preserve the one point space, and thus the 0th derivative is the value of the functor at a point, which is trivial. This makes the summands with 0th derivatives uninteresting. Thus on level one, μ boils down to a map $\partial_1 F \wedge \partial_1 G \to \partial_1(F \circ G)$.

We will focus on the impossibilities in level one, on the first derivatives. Recall that the first derivative is the linearization of the first cross effect, and linearization is a homotopy colimit

$$\operatorname{hocolim}\left(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \Omega^3\Sigma^3 X \to \cdots\right)$$

and so the map we desire must combine two of these homotopy colimits into one:

$$\operatorname*{hocolim}_k \Omega^k cr_1 F(S^k) \wedge \operatorname*{hocolim}_\ell \Omega^\ell cr_1 G(S^\ell) \to \operatorname*{hocolim}_n \Omega^n cr_1(F \circ G)(S^n)$$

This is a classical problem in homotopy theory, exactly the roadblock to finding a good category of spectra with a strictly associative smash product and also the cause of delay in defining THH. One solution to this problem, introduced in one form by Bökstedt in the 80s and in another by Smith in the 90s, is to include more symmetry, a technique we can employ by changing the shape of the homotopy colimit. Because of the recurring nature of this issue and resolution, we deem this the major key idea of the project.

Key Idea 1.3. *Including symmetry into a straight line homotopy colimit allows for a strictly associative combination.*

The current shape is indexed by the category $\mathbb{N}$ of finite sets $\underline{n} = \{1, 2, \ldots, n\}$ with standard inclusions $\{1, \ldots, n\} \hookrightarrow \{1, \ldots, n, n+1\}$

$$\emptyset \longrightarrow \underline{1} \longrightarrow \underline{2} \longrightarrow \underline{3} \longrightarrow \cdots$$

A new shape we could use is indexed by the category $\mathbb{I}$ of finite sets $\underline{n}$ with all injective maps

$$\begin{tikzcd} \emptyset \arrow[r] & \underline{1} \arrow[r] & \underline{2} \arrow[r] \arrow[loop above, "\Sigma_2"] & \underline{3} \arrow[r] \arrow[loop above, "\Sigma_3"] & \cdots \end{tikzcd}$$

This category has a symmetric monoidal product given by disjoint union of sets which yields a map on homotopy colimits

$$\operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}} G(U,V) \to \operatorname*{hocolim}_{U\amalg V\in\mathbb{I}} G(U \amalg V).$$

A similar map exists for the category $\mathbb{N}$, given by addition, but this does not translate to a strictly associative map on loops (which is why Moore loops were invented).

Indeed, linearization fits into the $\mathbb{I}$ diagram shape because $\Omega^n \Sigma^n$ has a natural Σ_n-action given by permuting the sphere coordinates. Thus, two homotopy colimits over $\mathbb{I}$ can be (strictly!) associatively reindexed to a single homotopy colimit over $\mathbb{I}$.

The next hurdle is that the cross effects must be combined. That is, the cross effects functor

$$cr_* : \mathrm{Fun}(\mathcal{T}, \mathcal{T}) \to \mathrm{Fun}(\Sigma, \mathrm{Fun}(\mathcal{T}^*, \mathcal{T}))$$

needs to be monoidal also. Again, this requires strict associativity, and cross effects are defined as total homotopy fibers, so the key to proving this is by careful choice of models, a topic which receives significant airtime in [**Yea17**]. We end up finding that induction works here, so another key idea is the following.

Key Idea 1.4. *A total homotopy fiber of a cube can be computed as an iterated homotopy fiber.*

This is well-known [**Goo91**], but in the case of the cross effects, we must rewrite a total homotopy fiber (an nth cross effect) as an iteration of very particular homotopy fibers: the first cross effects of some functor.

Combining the cross effects in this monoidal way also requires them to have assembly maps in each variable, a detail which is crucial to the models. A functor F has *assembly* if there are maps $F(X) \wedge Y \to F(X \wedge Y)$ for all spaces X, Y. We restrict our attention to *continuous* functors, which have assembly maps that also carry through to the cross effects. Continuity is the condition that there is a continuous map of spaces $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$, and so the functor F must be pointed (that is, $F(*) = *$). In the category of simplicial sets, a reduced functor ($F(*) \simeq *$) can be replaced functorially with a pointed functor.

The final hurdle we will discuss arises when looking at the higher level maps, for example, $\mu : \partial_1 F \wedge \partial_2 G \to \partial_2(F \circ G)$. The map μ must be equivariant with respect to the Σ_2-actions while still being associative. We will talk about the solution of this problem in Topic 2, but suffice it for now to say

Key Idea 1.5. *The sphere operad of* [**AK14**] *is awesome.*

Essentially, in order for the homotopy colimit combination to be effective for the single hocolim of $\partial_1 F$ with the double hocolim of $\partial_2 G$, we need to double up the sphere coordinates that we are linearizing along in $\partial_1 F$. This requires maps like $\Omega^k S^k \to \Omega^{2k} S^{2k}$, which can be easily defined as suspending by a k-sphere. For the equivariance, we need to choose a perpendicular complement of the existing k-spheres S^k and suspend in that direction. For the associativity to still hold, we need a really good choice of sphere complement, and the sphere operad is exactly that. We'll discuss how to picture this in Topic 2.

References

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Department of Mathematics, University of Maryland, College Park, MD 20742

E-mail address: `syeakel@math.umd.edu`