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# A user's guide: The slices of $S^n \wedge H\underline{\mathbb{Z}}$ for cyclic $p$-groups

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## 3. Story of the development

**3.1. The background.** In this section, we look at a brief history behind the problem of determining particular slice towers. Hill, Hopkins, and Ravenel were the first to fully develop and use the notion of the slice filtration as an expansion on the work of Dugger in [**Dug05**]. In [**HHR09**] they presented a formal definition of the filtration and used the particular slices of spectra built out of the spectrum MU in their solution to the Kervaire invariant one problem. While the towers of such spectra were determined rather straightforwardly, the towers of other, even seemingly simpler spectra, can be quite a bit more complicated. Additionally, there is still much unknown about the way the slice filtration filters spectra in general. Thus, one would like to know towers for a variety of spectra to get a better handle on what exactly the slice filtration does to $G$-spectra.

There are two aspects to consider when choosing the spectra whose towers we will compute. One is the group whose action we are considering. Beginning with cyclic $p$-groups is a logical place to start as the associated Mackey functors are relatively simple since all subgroups are nested. The other is of course the spectrum itself and suspensions of it by representation spheres. A natural place to begin on this front could be Eilenberg-MacLane spectra and their suspensions. $H\underline{\mathbb{Z}}$ is a classic choice and Hill, Hopkins, and Ravenel express in [**HHR15**] a goal of determining the slice towers of suspensions of $H\underline{\mathbb{Z}}$ by virtual representation spheres. In [**HHR15**], they compute the towers for particular suspensions by subrepresentations of the regular representation associated to cyclic $p$-groups. In a sense, the paper [**Yar15**] is a complementary one in that it determines the towers for suspensions of $H\underline{\mathbb{Z}}$ by trivial representations.

**3.2. The process.** While much of the early work on this project was essential to determining the final result, these computations play no part in the actual proof and thus all discussions concerning this part of the process were omitted from the paper. Here, I will lay out the early ideas and useful notions that ultimately lead to the main result.

In beginning to compute the slice tower for any given $G$-spectrum, there are a few results concerning the slice filtration in general that give a good starting point. One such result, given as Corollary 2.12 in [**Hil12**], provides the form of the $(-1)$-layer of any slice tower.

Useful Result 3.1. *The $(-1)$-layer of any $G$-spectrum $X$ is the $(-1)$-Postnikov layer:*

$$P_{-1}^{-1}(X) \simeq \Sigma^{-1} H\underline{\pi_{-1}(X)}$$

Though it is not true in general that all slices are integer suspensions of Eilenberg-MacLane spectra, we can still use this result to compute higher dimensional slices when used in conjunction with a second result concerning suspensions of the slice filtration.

We know that we cannot suspend a slice by an integral amount and necessarily get a slice in a higher dimension as in the Postinikov tower, but the slice tower does commute with suspensions by regular representation spheres. This result is again given by Hill in [**Hil12**].

Useful Result 3.2. *For any $G$-spectrum $X$, any $m$, and any $k$ we have:*

$$P^{m|G|+k}(\Sigma^{m\rho_G} X) = \Sigma^{m\rho_G} P^k(X)$$

*and thus,*

$$P_{m|G|+k}^{m|G|+k}(\Sigma^{m\rho_G} X) = \Sigma^{m\rho_G} P_k^k(X)$$

If in the above result we replace $X$ by $\Sigma^{-m\rho_G} X$, let $k = -1$ and apply result 3.1 we get the following:

Corollary 3.3. For any $G$-spectrum $X$ and any $m$ we have:

$$P_{m|G|-1}^{m|G|-1}(X) \simeq \Sigma^{m\rho_G - 1} H\underline{\pi_{-1}(\Sigma^{-m\rho_G} X)}$$

This combination of these two results provides a useful trick for computing all $(m|G|-1)$-slices of any $G$-spectrum $X$ as all one really needs to compute is the $(-1)$-homotopy of $\Sigma^{-m\rho_G} X$. In our case, we consider $X = S^n \wedge H\underline{\mathbb{Z}}$ and $G = C_{p^k}$, and thus need only determine $\underline{\pi_{-1}(S^{n-m\rho_{p^k}} \wedge H\underline{\mathbb{Z}})}$. This can be computed rather straightforwardly as the Bredon homology $\underline{H_{-1}(S^{n-m\rho_{p^k}}; \underline{\mathbb{Z}})}$ using chain complexes of Mackey functors, to obtain all $(mp^k - 1)$-slices.

Theorem 3.4. *For all integers $n \geq 0$ we have the following slices for the $C_{p^k}$-spectrum $S^n \wedge H\underline{\mathbb{Z}}$:*

$$P_{mp^k-1}^{mp^k-1}(S^n \wedge H\underline{\mathbb{Z}}) = \begin{cases} \Sigma^{m\rho_G - 1} H\underline{B}_{k,j} & \dfrac{n + 2\cdot 0^j}{p^{k-j}} \leq m \leq \dfrac{n-2}{p^{k-j-1}},\ m, n \text{ of same parity} \\ \Sigma^{m\rho_G - 1} H\underline{\mathbb{Z}}^* & n = rp^k - 1 \text{ and } m = \frac{n+1}{p^k} \\ * & \text{otherwise} \end{cases}$$

While this may look a bit daunting, if we examine the spectra themselves, $\Sigma^{m\rho_G - 1} H\underline{B}_{k,j}$ and $\Sigma^{m\rho_G - 1} H\underline{\mathbb{Z}}^*$ are exactly the spectra given in dimensions $mp^k - 1$ from our Key Idea 1.2. We now know of course that these are not all of the nontrivial slices of $S^n \wedge H\underline{\mathbb{Z}}$ but it gives a good framework upon which to build. The most obvious next question is then, "In what dimensions aside from $mp^k - 1$ are there nontrivial slices?" At this point, another result from [**Hil12**] concerning the slice filtration is useful. In the following, $H$ is any subgroup of $G$ and $i_H^*$ is the forgetful functor from $G$-spectra to $H$-spectra.

Useful Result 3.5. *The restriction to $H$ of the slice tower of $X$ is the slice tower of $X$ restricted to $H$:*

$$P^n i_H^*(X) = i_H^* P^n(X)$$

How does this help? Let us consider an example. Suppose we wish to compute the slice tower for the $C_{p^2}$-spectrum $S^n \wedge H\underline{\mathbb{Z}}$. From Theorem 3.4, we know that we have nontrivial slices in dimensions $mp^2 - 1$ for various $m$. However, upon restricting $S^n \wedge H\underline{\mathbb{Z}}$ to $C_p$, we know by Result 3.5 that we should also have nontivial slices in dimensions $mp - 1$ for specified $m$. Thus, we have determined another class of nontrivial slices that must appear in the tower. Extending this result we can easily see that for the $C_{p^k}$ spectrum $S^n \wedge H\underline{\mathbb{Z}}$ we should have nontrivial slices in dimensions $mp^d - 1$ for all $1 \leq d \leq k$.

It is not too difficult to see that the $mp - 1$ slices determined by such computations are in fact the only nontrivial slices for $S^n \wedge H\underline{\mathbb{Z}}$ considered as a $C_p$-spectrum. Hence, it is not too far a leap to guess that for the $C_{p^k}$ spectra the $(mp^d - 1)$-slices are in fact the only nontrivial ones.

## References

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